Justification for the PErpendicular Line Slope Theorem (PELST)

In class, we learned the following:

PELST: Two lines are perpendicular if and only if their slopes are opposite reciprocals of each other.

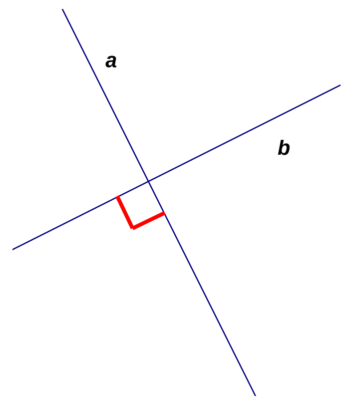
Now, that "if and only if" might throw you for a loop, but here's what it means:

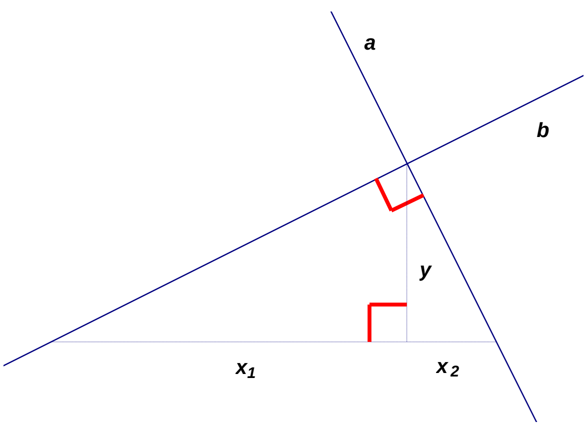
1) If two lines are perpendicular, then their slopes are opposite reciprocals (that is, their product is -1);
2) If two lines' slopes are opposite reciprocals, then they are perpendicular.

Technically, to prove PELST, we need to prove both of these. Let's do it!

Proof of 1): Assume the two lines are parallel. We can prove part 1 using a little bit of geometry, mixed with algebra. I've drawn a picture of two perpendicular lines, ***a*** and ***b***, at right. Notice that one slope is positive, and one negative. We will need this fact a little later.

To assist us in our studies, an idea from geometry called similarity is going to come in handy. Figures are similar if they have equal angles and proportional sides. Let's find some similar figures:





I've placed a couple of segments into the figure above to create the figure at right. Please notice that we have created three right triangles, all of which are similar to each other[1]. This means that we have the following relationship:

$$\frac{x_1}{y} = \frac{y}{x_2}$$

Now, we also know that the slope of line ***a*** is $\frac{y}{x_2}$, and the slope of line ***b*** is $\frac{y}{x_1}$.

It's time to put these facts together! Check this out:

$$\frac{x_1}{y} = \frac{y}{x_2}$$

[1] By the way, if you're the first person to prove ***why*** the three triangles are similar, I'll give you 5 extra credit points.

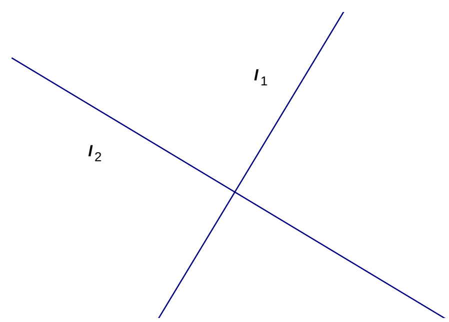
$$\frac{1}{\text{slope of line } \boldsymbol{b}} = \text{slope of line } \boldsymbol{a}$$

At this point, we have to remember that one of the two lines has a negative slope. Thus, we can conclude that

$$(\text{slope of line } \boldsymbol{a})(\text{slope of line } \boldsymbol{b}) = -1$$

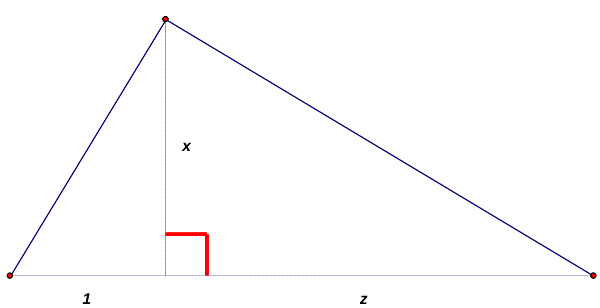
Which means the slope are negative reciprocals of each other. $\therefore$

Proof of 2): This part's gonna be a hoot. We have to assume that two lines have slopes that are negative reciprocals, and then derive their perpendicularity from that. Ha! Sounds like fun. Let's start with the diagram of two lines whose slopes are negative reciprocals[2]. Let's use subscript notation for these lines' slopes, as follows:



Slope of $\boldsymbol{l}_1 = \boldsymbol{m}_{l1} = \boldsymbol{x}$

Slope of $\boldsymbol{l}_2 = \boldsymbol{m}_{l2} = \dfrac{-1}{\boldsymbol{x}}$



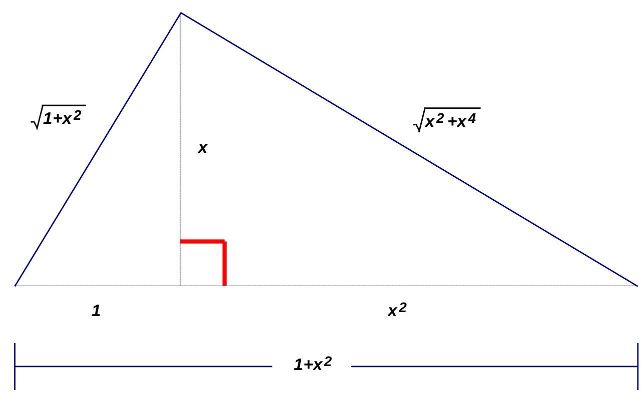
OK, I've slimmed down the diagram a bit, and added some things. The dotted lines represent horizontal and vertical segments, which allow us to "see" the slope. $\boldsymbol{l}_1$ is now represented by the left diagonal segment, and $\boldsymbol{l}_2$ the right segment. If we can show these segments are perpendicular, then the full lines will be perpendicular, as well.

Since the slope of $\boldsymbol{l}_1$ is $\boldsymbol{x}$, I've written the rise and run as $\boldsymbol{x}$ and 1, respectively. Now, $\boldsymbol{l}_2$ has a slope of $\dfrac{-1}{\boldsymbol{x}}$, but I've drawn its rise to be $\boldsymbol{x}$. Therefore, I needed to call its run something else (I chose $\boldsymbol{z}$). However, I know that $\dfrac{-1}{\boldsymbol{x}} = \dfrac{-\boldsymbol{x}}{\boldsymbol{z}}$ (the negative's there for the same reason as in the previous proof). Solving this for $\boldsymbol{z}$, we find that $\boldsymbol{z} = \boldsymbol{x}^2$.

[2] I know they look perpendicular in the diagram. Trust me, I'm not using the fact that they are perpendicular at all in the proof. That would be circular reasoning, which is bad.

Let's redraw our triangle, with a few more additions, courtesy of Pythagoras[3]. That's him at right.

Speaking of Pythagoras...were you aware of the ***converse***[4] of the Pythagorean Theorem? It states this: if the square of one side of a triangle equals the sum of the squares of the other two sides, the triangle has to be a right triangle. Let's try that on our big triangle's sides:



$$(1+\boldsymbol{x}^2) \stackrel{?}{=} (\sqrt{1+\boldsymbol{x}^2}\,)^2 + (\sqrt{\boldsymbol{x}^2+\boldsymbol{x}^4}\,)^2$$

$$1 + 2\boldsymbol{x}^2 + \boldsymbol{x}^4 \stackrel{?}{=} 1 + \boldsymbol{x}^2 + \boldsymbol{x}^2 + \boldsymbol{x}^4$$

$$1 + 2\boldsymbol{x}^2 + \boldsymbol{x}^4 = 1 + 2\boldsymbol{x}^2 + \boldsymbol{x}^4$$

Since we have found equality, we can conclude that the triangle is, in fact, right. Therefore, we can conclude that $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ must be perpendicular. $\therefore$

So, since we have proven both 1) and 2), we have proven PELST. Yahoo!

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[3] Actually, the Chinese had it before he did. How'd he get credit?

[4] If a statement is in the form "if a, then b", its converse is in the form "if b, then a". In proving PALST and PELST,. We had to prove both a statement and its converse...but both are not necessarily true. Take, for example, the statement, "If there is a cat in my gas tank, then my car won't start." Sounds pretty true. Now, consider the converse: "If my car won't start, then there's a cat in my gas tank." Hmmmm....